When V is a minimum, $dV/d\theta=0$, which condition is satisfied when $\sin\theta=\sqrt{\frac{1}{3}}$, *i. e.*, $V=\frac{1}{2}(\pi r^3\sqrt{3})$.

Also solved by R. A. Wells, J. E. Sanders, G. W. Greenwood, W. W. Landis, G. B. M. Zerr, E. L. Sherwood, J. Scheffer, and L. E. Newcomb.

MECHANICS.

167. Proposed by EDWIN S. CRAWLEY, Ph.D., Professor of Mathematics in the University of Pennsylvania.

An anchor ring or torus is produced by the revolution of a circle of radius r, the center of the revolving circle describing a circle of radius R. A quadrant of the torus is cut by two planes through the center of the ring perpendicular to each other and perpendicular to the plane of revolution. Determine the limiting value of the ratio $R:r$, so that when the quadrant thus formed is placed with one of its ends in coincidence with a horizontal plane it will rest in that position.

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

The center of gravity of the quadrant is at a distance d from the axis of revolution where $d=(4R^2+r^2)/\pi R\sqrt{2}$ [Routh's *Statics* (second edition), Vol. I, ex. 4, p. 280].* The limiting ratio of R to r is given by $d/\sqrt{2}=R-r$, whence $4R^2+r^2=2\pi R(R-r)$; and $R/r=2.9+$.

II. Solution by B. F. FINKEL, Fellow in Mathematics, University of Pennsylvania, Philadelphia, Pa.

The equation of the surface of the ring is $\rho=R\sin\phi\pm\sqrt{(r^2-R^2\cos^2\phi)}$, when r is the radius of a cross-section of the ring and R the radius of the path of the center of gravity of the generating circle. Then

$$\bar{x}=2.\frac{\int\sigma x dV}{\int dV},$$

where σ (=1, say) is the density dV and element of volume, and x the distance from the y-axis to this element of volume.

Then $dV=\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$,

$x=\rho\sin\phi\cos\theta$, and

$\int dV=2\pi R.\pi r/4=\frac{1}{4}\pi^2r^2R$. Hence

$$\bar{x}=\frac{4}{\pi^2r^2R}\int_{\cos^{-1}(r/R)}^{\frac{1}{2}\pi}\sin^2\phi\, d\phi\int_{R\sin\phi-\sqrt{(r^2-R^2\cos^2\phi)}}^{R\sin\phi+\sqrt{(r^2-R^2\cos^2\phi)}}\rho^3 d\rho\int_0^{\frac{1}{2}\pi}d\theta$$

$$=\frac{8}{\pi^2r^2}\int_{\cos^{-1}(r/R)}^{\frac{1}{2}\pi}[\sin^3\phi(2R^2\sin^2\phi+r^2-R^2)\sqrt{(r^2-R^2\cos^2\phi)}]d\phi.$$

Let $\cos\phi=(r/R)\cos\psi$; then the last equation reduces to

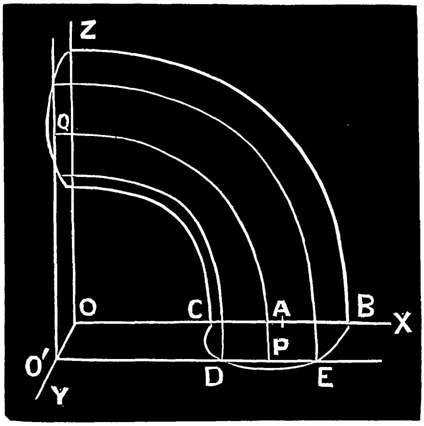
*See also Bowser's *Analytical Mechanics*, 9th ed., p. 112.

$$\bar{x}=\frac{8}{\pi^2 R}\int_0^{\frac{1}{2}\pi}[R^2+r^2-\frac{r^2}{R^2}(3R^2+r^2)\cos^2\psi+2\frac{r^2}{R^2}\cos^4\psi]\sin^2\psi d\psi=\frac{4R^2+r^2}{2\pi R}$$

To insure stability, this must be greater than $R-r$. Hence, the limiting ratio is

$$\frac{R}{r^2}=\frac{\pi+\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)}=2.9+.$$

III. Solution by the PROPOSER.



$OA=R$, $AB=AE=r$. Equation of section $BEDC$ is

$$(x-R)^2+y^2=r^2.$$

$$\therefore O'E=R+\sqrt{(r^2-y^2)},$$
$$O'D=R-\sqrt{(r^2-y^2)}.$$

Take as element of volume the thread PQ whose length is $\frac{1}{2}\pi x$ and cross-section $dxdy$. Then, if OY be taken as axis of moments the abscissa of the center of gravity of PQ is $2x/\pi$. Hence, the moment of the attraction of gravity upon the whole solid is

$$2\int_0^r\int_{R-\sqrt{(r^2-y^2)}}^{R+\sqrt{(r^2-y^2)}}x^2dxdy=4R^2\int_0^r\sqrt{(r^2-y^2)}dy+\tfrac{4}{3}\int_0^r(r^2-y^2)^{\frac{3}{2}}dy$$

$$=4r^2R^2\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta+\tfrac{4}{3}r^4\int_0^{\frac{1}{2}\pi}\cos^4\theta d\theta=\pi r^2R^2+\tfrac{1}{4}\pi r^4=\frac{\pi r^2}{4}(r^2+4R^2).$$

The volume of the solid is $\frac{1}{4}\pi r^2.2\pi R=\frac{1}{2}\pi^2r^2R$. Hence, the arm of the center of gravity is

$$\frac{\pi r^2}{4}(r^2+4R^2)\div\tfrac{1}{2}\pi^2r^2R=\frac{r^2+4R^2}{2\pi R}$$

If the solid is about to topple over the center of gravity is above C, and $\frac{r^2+4R^2}{2\pi R}=R-r$, from which

$$\frac{R}{r}=\frac{\pi\pm\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)},$$ or discarding the negative result,

$$\frac{R}{r}=\frac{\pi+\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)}=2.902,$$ approximately.

Also solved by S. A. Corey, Hiteman, Iowa.